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POWER CHAINS IN A DIVISOR GRAPH

Satyanarayana Bhavanari[1], Srinivasulu Devanaboina[2], Mallikarjuna Bhavanari[3] & Abul Basar[4]

[1]*Department of Mathematics, Acharya Nagarjuna University, Andhra Pradesh, India*

[2]*Department of FEd, NRI Institute of Technology, Andhra Pradesh, India*

[3]*Department of Mechanical Engineering, Institute of Energy Engineering, National Central University, Jhongli, Taoyuan, Taiwan*

[4]*Department of Natural and Applied Science, Glocal University, Uttar Pradesh, India*

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ABSTRACT

The divisor graph of an associative ring R (denoted as DG(R)) was introduced by Satyanarayana, Srinivasulu.[9]. In this paper, we introduce a simple concept "Power Chain in a Divisor Graph" .We prove that if $0 \neq a \in R$ is nilpotent, then the power chain starting with a is of finite length. If DG(R) (the divisor graph of R) contains a power chain starting with $a \in R$ which is of infinite length, then $0 \neq a \neq 1$, a is non–idempotent and non–nilpotent element. We announce some basic results. Finally, we deduce that if R be an integral domain and a $\in R$, then $0 \neq a \neq 1$ if and only if the power chain starting with a (in DG(R)) is of infinite length.

KEYWORDS: *Associative Ring, Divisor Graph of a Ring, Complete Graph*

***Mathematics Subject Classification**: 05C07, 05C20, 05C76, 05C99, 13E15*

1. INTRODUCTION

Beck [2] related a commutative ring R to a graph by using the elements of R as vertices and two vertices x, y are adjacent if and only if $xy = 0$. Anderson and Livingston [1] proposed a modified method of associating a commutative ring to a graph by introducing the concept of a zero-divisor graph of a commutative ring. Satyanarayana Bhavanari, Syam Prasad K and Nagaraju D [26] introduced "Prime Graph" of a ring and later studied by several authors. These concepts are different bridges connecting the two theories: Ring Theory & Graph Theory.

Now we introduce a concept called "Power Chains in a divisor graph" of a ring. This idea motivates us to prove the following results: (i) DG($\mathbb{Z}_n$) contains a chain of length $\varphi(n) - 1$. (ii) If p-prime , then DG($\mathbb{Z}_p$) contain a max chain of length p -2.

Now we review some definitions and results for the sake of completeness.

1.1 Definitions

Let $G = (V(G), E(G))$ be a graph where $V(G)$ is the set of vertices of G and $E(G)$ the set of edges of G. An edge between two vertices $x, y \in V(G)$ is denoted by $\overline{xy}$.

- A graph G (V, E) is said to be a star graph if there exists a fixed vertex v such that E = {vu / u $\in$ V and u $\neq$ v}. A star graph is said to be an n-star graph if the number of vertices of the graph is n.
- (Satyanarayana, Srinivasulu D &Mallikarjuna [14]): Let G be a graph. The star number of G is defined as max {n / there exists an n-star graph which is a subgraph of G and n is an integer with $n \geq 1$}. We denote this star number of G by $S_n(G)$.
- (Satyanarayana Bhavanari and Syam Prasad K [25]) A complete graph is a simple graph in which each pair of distinct vertices are joined by an edge. The complete graph on 'n' vertices is denoted by K_n.
- **(**Satyanarayana Bhavanari, Srinivasulu Devanaboina, AbulBasar & Mallikarjuna Bhavanari [9]) Let R be an associative ring and $x, y \in R$. We say that x divides y (if there exists $z \in R$ such that $xz = y$ or $zx = y$).A graph G = (V, E) is said to be the **divisor graph** of R (denoted by DG(R)) if V = R and $E = \{\overline{xy} / xz = y \text{ or } zx = y \text{ for some } z \in R \text{ and } x \neq y\}$.

Power Chains in a Divisor Graph

2.1. Definition

A chain

Figure 1

is said to be a power chain starting with a if x_1= a and x_n = a.$(x_{n-1})$and$x_{n-1} \neq x_n$ for all $n \geq 1$.

2.2 Note: If a$\in$ R is an idempotent then $a = a^2$ and so there is no edge in DG(R)) between a and a^2.

2.3 Examples: If R = $\mathbb{Z}_2$ = {0, 1} the ring of integers modulo 2, then V (DG(R)) = {0, 1}. E (DG(R)) = $\{\overline{01}\}$. Now DG(R) is given in Figure 2.

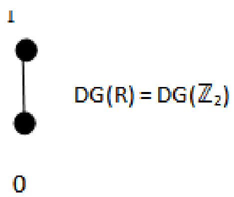


Figure 2

If R = $\mathbb{Z}_3$ = {0, 1, 2} the ring of integers modulo 3, V (DG(R)) = {0, 1, 2} and E(DG(R)) = { $\overline{01}$, $\overline{02}$, $\overline{12}$ }. Now there is only one power chain in DG(R) and it is given in Figure 3.

Figure 3

If R = $\mathbb{Z}_4$ = {0, 1, 2, 3} the ring of integers modulo 4, V(DG(R)) = {0, 1, 2, 3} and E(DG(R))= {$\overline{01}$, $\overline{02}$, $\overline{03}$, $\overline{12}$, $\overline{23}$, $\overline{13}$ }. Now there exist two power chains in DG(R) and are given in Figure 4.

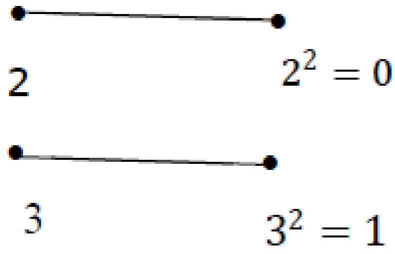


Figure 4

If $R = \mathbb{Z}_5$, then R = $\mathbb{Z}_5$ = {0, 1, 2, 3, 4} the ring of integers modulo 5, V(DG(R)) = {0, 1, 2, 3, 4} and E(DG(R))= {$\overline{01}$, $\overline{02}$, $\overline{03}$, $\overline{04}$, $\overline{12}$, $\overline{13}$, $\overline{14}$, $\overline{23}$, $\overline{24}$, $\overline{34}$}. Now power chains in DG(R) is given in Figure 5.

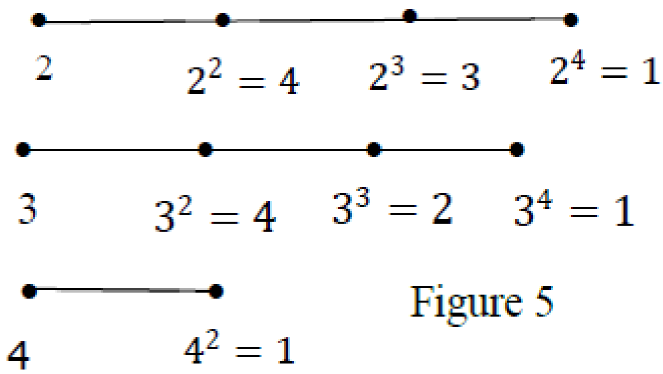


Figure 5

If $R = \mathbb{Z}_6$, then R = $\mathbb{Z}_6$ = {0, 1, 2, 3, 4, 5} the ring of integers modulo 6, V(DG(R)) = {0, 1, 2, 3, 4, 5} and E(DG(R))= {$\overline{01}$, $\overline{02}$, $\overline{03}$, $\overline{04}$, $\overline{05}$, $\overline{12}$, $\overline{13}$, $\overline{14}$, $\overline{15}$, $\overline{23}$, $\overline{24}$, $\overline{34}$}. Now Power chains inDG(R) is given in Figure 6.

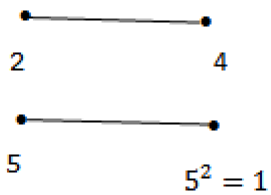


Figure 6

If $R = \mathbb{Z}_7$, then R = $\mathbb{Z}_7$ = {0, 1, 2, 3, 4, 5, 6} the ring of integers modulo 7, V(DG(R)) = {0, 1, 2, 3, 4, 5, 6} and E(DG(R))= {$\overline{01}$, $\overline{02}$, $\overline{03}$, $\overline{04}$, $\overline{05}$, $\overline{06}$, $\overline{12}$, $\overline{13}$, $\overline{14}$, $\overline{15}$, $\overline{16}$, $\overline{23}$, $\overline{24}$, $\overline{25}$, $\overline{26}$, $\overline{34}$, $\overline{35}$, $\overline{36}$, $\overline{45}$, $\overline{46}$, $\overline{56}$}. Now Power chains in DG(R) is given in Figure 7.

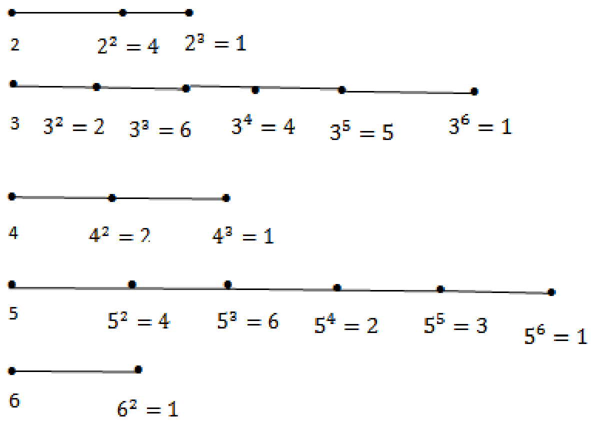


Figure 7

2.4. Results

- DG($\mathbb{Z}_n$) contains a chain of length $\varphi(n) - 1$
- **(ii)**If p-prime , then DG($\mathbb{Z}_p$) contain a max chain of length p -2

2.5 Lemma: If $0 \neq a \in R$ is nilpotent then the power chain starting with a is of finite length.

Proof: Suppose that $a \in R$ is a nilpotent element. Then there exists a positive integer k such that $a^k = 0$. Let m be the least positive integer such that $a^m = 0$. Now write x_1= a, x_n = a.(x_{n-1}).

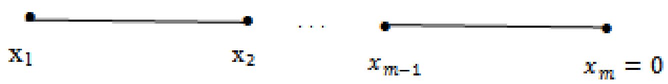
Now 

Figure 8

is the power chain starting with ‘a $\in R$ and its length is m, a finite length.

2.6 Lemma: If DG (R) contains a power chain starting with a $\in R$ which is of infinite length, then $0 \neq a \neq 1$, a is non–idempotent and non – nilpotent element.

Proof: Suppose that DG(R) contains a power chain starting with a which is of infinite length. Suppose the chain is

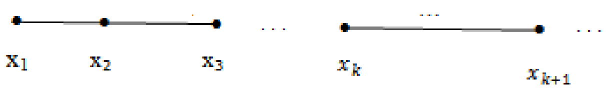


Figure 9

withx_1= a and x_n = a.(x_{n-1}) = a^n, $x_{n-1} \neq x_n$ for all n.

Since $x_1 \neq x_2$ we have that $a \neq a^2$ and so a is not idempotent.

If a = 0 then x_1= 0 =x_2, a contradiction.

Suppose a is the nilpotent element. Then by above lemma, the power chain starting with a is of finite length, a contradiction.

Therefore a con not be a nilpotent element.

2.7 Lemma: Let R be an integral domain. If $0 \neq a \in R$ then a cannot be a nilpotent element.

Proof: Suppose a is nilpotent, Then there exists a positive integer such that$a^n = 0$ without loss of generality we assume that n is the least positive integer such that $a^n = 0$. Now a $.(a^{n-1}) = 0$ and $a \neq 0, a^{n-1} \neq 0$, a contradiction. The proof is complete.

2.8. Theorem

Let R be an integral domain and a $\in R$. Then $0 \neq a \neq 1$if and only if the power chain starting with a (inDG(R)) is of infinite length.

Proof: Suppose a is non-zero element in R.

Then $a^k \neq 0$ for any positive integer. (by lemma – 2.7)

Now we prove that $a^k \neq a^{k+1}$ for all k ≥1. Suppose a^k= a^{k+1}. Then $a^k(1-a)$ = 0 ⇒ $(1-a)$ = 0 ($since\ a^k \neq 0$)

$\Rightarrow a = 1$, a contradiction.

This shows that $a^k \neq a^{k+1}$ for any positive integer k.

So the edge $\overline{a^k a^{k+1}}$ is in the divisor graph DG(R). This is true for all positive integers k.

Therefore the chain given here.

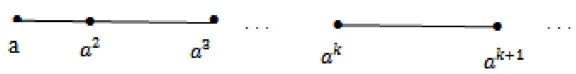


Figure 10

(that is the power chain starting with a) is an infinite chain.

Now the converse follows from Lemma 2.6.

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